On the moment formulas for the noncentral Wishart distributions

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Contents

1 Moment formulas for the noncentral Wishart distributions

Real Wishart distribution $W_p(\alpha, \Sigma, \Delta)$

- $W = (w_{ij})$: $p \times p$ symmetric n.n.d. random matrix with mgf (moment generating func)

$$\begin{aligned}\phi(\Theta) &= E\left[e^{\operatorname{tr}(\Theta W)}\right] \\ &= \det(I - 2\Theta\Sigma)^{-\frac{\alpha}{2}} e^{\operatorname{tr}(I-2\Theta\Sigma)^{-1}\Theta\Delta}\end{aligned}$$

α : degrees of freedom
$\Sigma = (\sigma_{ij})$: covariance matrix
$\Delta = (\delta_{ij})$: mean square matrix
$\Sigma^{-1}\Delta$: noncentrality matrix

Complex Wishart distribution $CW_p(\alpha, \widetilde{\Sigma}, \widetilde{\Delta})$

- $\widetilde{W} = (\widetilde{w}_{ij})$: $p \times p$ Hermitian n.n.d. random matrix with mgf

$$\widetilde{\phi}(\Theta) = E\left[e^{\mathrm{tr}(\Theta \widetilde{W})}\right]$$
$$= \det(I - \Theta\widetilde{\Sigma})^{-\alpha} e^{\mathrm{tr}(I - \Theta\widetilde{\Sigma})^{-1}\Theta\widetilde{\Delta}}$$

α : degrees of freedom

$\widetilde{\Sigma} = (\widetilde{\sigma}_{ij})$: covariance matrix

$\widetilde{\Delta} = (\widetilde{\delta}_{ij})$: mean square matrix

$\widetilde{\Sigma}^{-1}\widetilde{\Delta}$: noncentrality matrix

Purpose

- To evaluate the moments $E[w_{i_1 i_2} \cdots w_{i_{2n-1} i_{2n}}]$, i.e., the Taylor expansion coefficients of the mgf $\phi(\Theta)$

- Remark: Wlog, we can assume that $i_1, i_2, \ldots, i_{2n}$ are different by considering "degenerate" Wishart distribution

Moment formula (complex case)

- Example: $(n = 3,\ \bar{i} = i + n)$

$$\begin{aligned} E[w_{1\bar{1}}w_{2\bar{2}}w_{3\bar{3}}] = {} & \alpha^3\sigma_{1\bar{1}}\sigma_{2\bar{2}}\sigma_{3\bar{3}} + \alpha^2\sigma_{1\bar{2}}\sigma_{2\bar{1}}\sigma_{3\bar{3}}[3] \\ & + \alpha\sigma_{1\bar{2}}\sigma_{2\bar{3}}\sigma_{3\bar{1}}[2] + \alpha^2\sigma_{1\bar{1}}\sigma_{2\bar{2}}\delta_{3\bar{3}}[3] \\ & + \alpha\sigma_{1\bar{2}}\sigma_{2\bar{1}}\delta_{3\bar{3}}[3] + \alpha\sigma_{1\bar{1}}\sigma_{2\bar{3}}\delta_{3\bar{2}}[6] + \sigma_{1\bar{2}}\sigma_{2\bar{3}}\delta_{3\bar{1}}[6] \\ & + \alpha\sigma_{1\bar{1}}\delta_{2\bar{2}}\delta_{3\bar{3}}[3] + \sigma_{1\bar{2}}\delta_{2\bar{1}}\delta_{3\bar{3}}[6] \\ & + \delta_{1\bar{1}}\delta_{2\bar{2}}\delta_{3\bar{3}} \end{aligned}$$

- Each term is interpreted using graph terminology with vertices

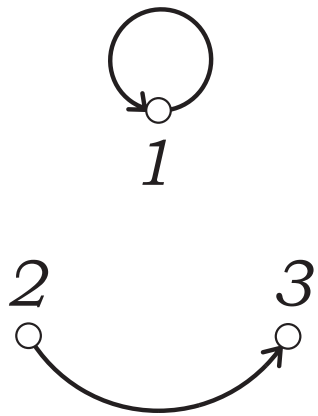
$$V = \{1, 2, 3\}$$

Let

$$\Pi = \{\pi : V_1 \to V \text{ (injection)} \mid V_1 \subset V\}$$

For each $\pi \in \Pi$**, e.g.,** $\pi : 1 \mapsto 1,\ 2 \mapsto 3$**, define a directed graph** $G_\pi = (V, E_\pi)$

$$E_\pi = \{(v, \pi(v))\} = \{(1, 1), (2, 3)\}$$



G_π consists of (directed) cycles and chains

$$\mathrm{len}(G_\pi) = \#\text{ of cycles} = 1$$

$$\begin{aligned}\check{E}_\pi = &\{(v_1, v_2) \mid \text{ ordered pair of terminating and} \\ &\qquad\qquad \text{starting vertices of a chain}\} \\ = &\{(3, 2)\}\end{aligned}$$

This graph G_π represents the term

$$\alpha^1 \sigma_{1\bar{1}} \sigma_{2\bar{3}} \delta_{3\bar{2}}$$

- Theorem: Let $(\widetilde{w}_{ij}) \sim CW_p(\alpha, \widetilde{\Sigma}, \widetilde{\Delta})$

$$\begin{aligned} E[\widetilde{w}_{1\bar{1}} \cdots \widetilde{w}_{n\bar{n}}] &= \sum_{\pi} \alpha^{\mathrm{len}(G_\pi)} \prod_{(u,v)\in E_\pi} \widetilde{\sigma}_{i\bar{j}} \prod_{(i,j)\in \check{E}_\pi} \widetilde{\delta}_{i\bar{j}} \\ &= \widetilde{\Xi}_n(\alpha; (\widetilde{\sigma}_{i\bar{j}}), (\widetilde{\delta}_{i\bar{j}})), \text{ say} \end{aligned}$$

- Remark: When $\widetilde{\Delta} = 0$, $\alpha^n \widetilde{\Xi}_n(\alpha^{-1}; (\widetilde{\sigma}_{i\bar{j}}), 0)$ is α-permanent of $(\widetilde{\sigma}_{i\bar{j}})$ (Vere-Jones)

Moment formula (real case)

- Example:

$$E[w_{12}w_{34}w_{56}] = \alpha^3\sigma_{12}\sigma_{34}\sigma_{56} + \alpha^2\sigma_{23}\sigma_{14}\sigma_{56}[6]$$
$$+ \alpha\sigma_{23}\sigma_{45}\sigma_{16}[8] + \alpha^2\sigma_{12}\sigma_{34}\delta_{56}[3] + \alpha\sigma_{23}\sigma_{14}\delta_{56}[6]$$
$$+ \alpha\sigma_{12}\sigma_{45}\delta_{36}[12] + \sigma_{23}\sigma_{45}\delta_{16}[24]$$
$$+ \alpha\sigma_{12}\delta_{34}\delta_{56}[3] + \sigma_{23}\delta_{14}\delta_{56}[12] + \delta_{12}\delta_{34}\delta_{56}$$

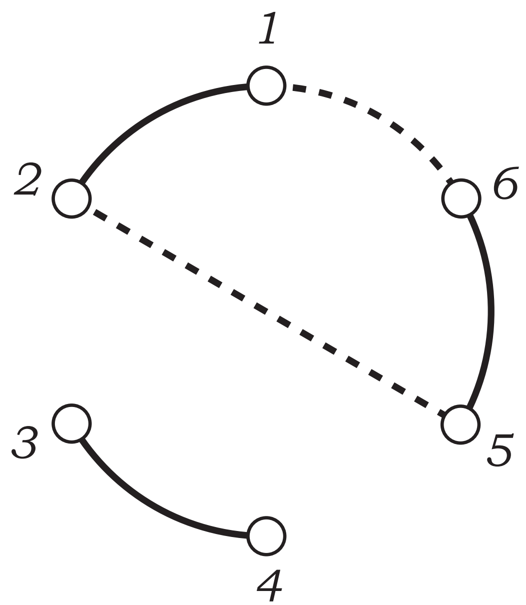
- Graph presentation:

$$V = \{1, 2, 3, 4, 5, 6\}$$

$$E_0 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$$

$$\mathcal{E} = \{\text{unordered perfect matching in } V_1 \mid V_1 \subset V\}$$

For each $\boldsymbol{E} \in \boldsymbol{\mathcal{E}}$, say $\boldsymbol{E} = \{\{1,6\},\{2,5\}\}$,
define an undirected graph $\boldsymbol{G} = (\boldsymbol{V}, \boldsymbol{E}_0 \cup \boldsymbol{E})$



$\mathbf{len}(\boldsymbol{G}) = \#$ **of cycles** $= \mathbf{1}$

$$\check{E} = \{(v_1, v_2) \mid \text{unordered pair of terminating vertices of a chain}\} = \{(3,4)\}$$

This graph represents the term

$$\alpha^1 \sigma_{16} \sigma_{25} \delta_{34}$$

- Theorem: Let $(w_{ij}) \sim W_p(\alpha, \Sigma, \Delta)$

$$E[w_{12} \cdots w_{2n-12n}] = \sum_{E} \alpha^{\text{len}(G)} \prod_{(u,v)\in E} \sigma_{ij} \prod_{(i,j)\in \check{E}} \delta_{ij}$$
$$= \Xi_n(\alpha; (\sigma_{ij}), (\delta_{ij})), \text{ say}$$

- Remark: When $\Delta = 0$, $\Xi_n(1; (\widetilde{\sigma}_{ij}), 0)$ is a Hafnian of (σ_{ij})

Literature

- **Central cases**
  - **Takemura (1991) (in Japanese)**
  - **Lu and Richards (2001)**
  - **Graczyk, Letac and Massam (2003, 2005)**
- **Noncentral cases**
  - **Letac and Massam (2008)**
  - **Kuriki and Numata (2010) AISM**

2 An application: Infinite divisibility of noncentral multivariate gamma distribution

Noncentral multivariate gamma distribution

- Definition:

$$(\xi_1, \ldots, \xi_p) = \frac{1}{2}(w_{11}, \ldots, w_{pp}) \sim MG_p(\alpha, \Sigma, \Delta)$$

where $(w_{ij}) \sim W_p(2\alpha, \Sigma, 2\Delta)$

- Mgf:

$$\phi(\theta) = E\left[e^{\sum \theta_i \xi_i}\right] = \det(I - \Theta\Sigma)^{-\alpha} e^{\mathrm{tr}(I-\Theta\Sigma)^{-1}\Theta\Delta}$$

with $\Theta = \mathrm{diag}(\theta)$

- Moment:

$$E[\xi_1 \cdots \xi_n] = \widetilde{\Xi}_n(\alpha; (\sigma_{ij}), (\delta_{ij}))$$

Problem

- **When is $MG_p(\alpha, \Sigma, \Delta)$ infinitely divisible, i.e., "$\phi(\theta)^{1/k}$ is a mgf of prob measure for all k"?**

- **Infinite divisibility of MG has been attracting attention historically, e.g., Griffiths (1984), Bapat (1989), Vere-Jones (1967, 1997), Bernardoff (2006), etc.**

Theorem

- Assume that $\Sigma^{-1} = (\sigma^{ij})$ exists. $MG_p(\alpha, \Sigma, \beta\Delta)$ exists for all small $\alpha > 0$ and $\beta \geq 0$ iff for all $\{i_1, \ldots, i_k\} \subset \{1, \ldots, p\}$,

$$(-1)^k \sigma^{i_1 i_2} \sigma^{i_2 i_3} \ldots \sigma^{i_k i_1} \geq 0$$

$$(-1)^{k-1} \sigma^{i_1 i_2} \sigma^{i_2 i_3} \ldots \sigma^{i_{k-1} i_k} \delta^{i_k i_1} \geq 0$$

  hold where $\delta^{ij} = (\Sigma^{-1} \Delta \Sigma^{-1})_{ij}$

- Corollary: Letting $\alpha = \beta = 1/k$, $MG_p(1, \Sigma, \Delta)$ is shown to be infinitely divisible under the conditions of Thm.

Griffiths' idea

- From r.v. $X \geq 0$ with mgf $\phi(\theta) = E[e^{\theta X}]$, we can define a discrete probability with the pgf

$$\psi_b(z) := \phi(b(z-1)) = E[e^{b(z-1)}] = \sum p_k z^k$$

where $p_k = \frac{b^k}{k!} E[X^k e^{-bX}]$ $(b > 0)$

- Conversely if $\psi_b(z) = \phi(b(z-1))$ a pgf of r.v. Y on $\mathbb{Z}_{\geq 0}$, then as $b \to \infty$, $Y/b \Rightarrow X$ with mgf ϕ because of

$$\lim_{b \to \infty} \psi_b(e^{\theta/b}) = \phi(\theta)$$

and the continuity thm for Laplace transform

Proof

- In our problem,

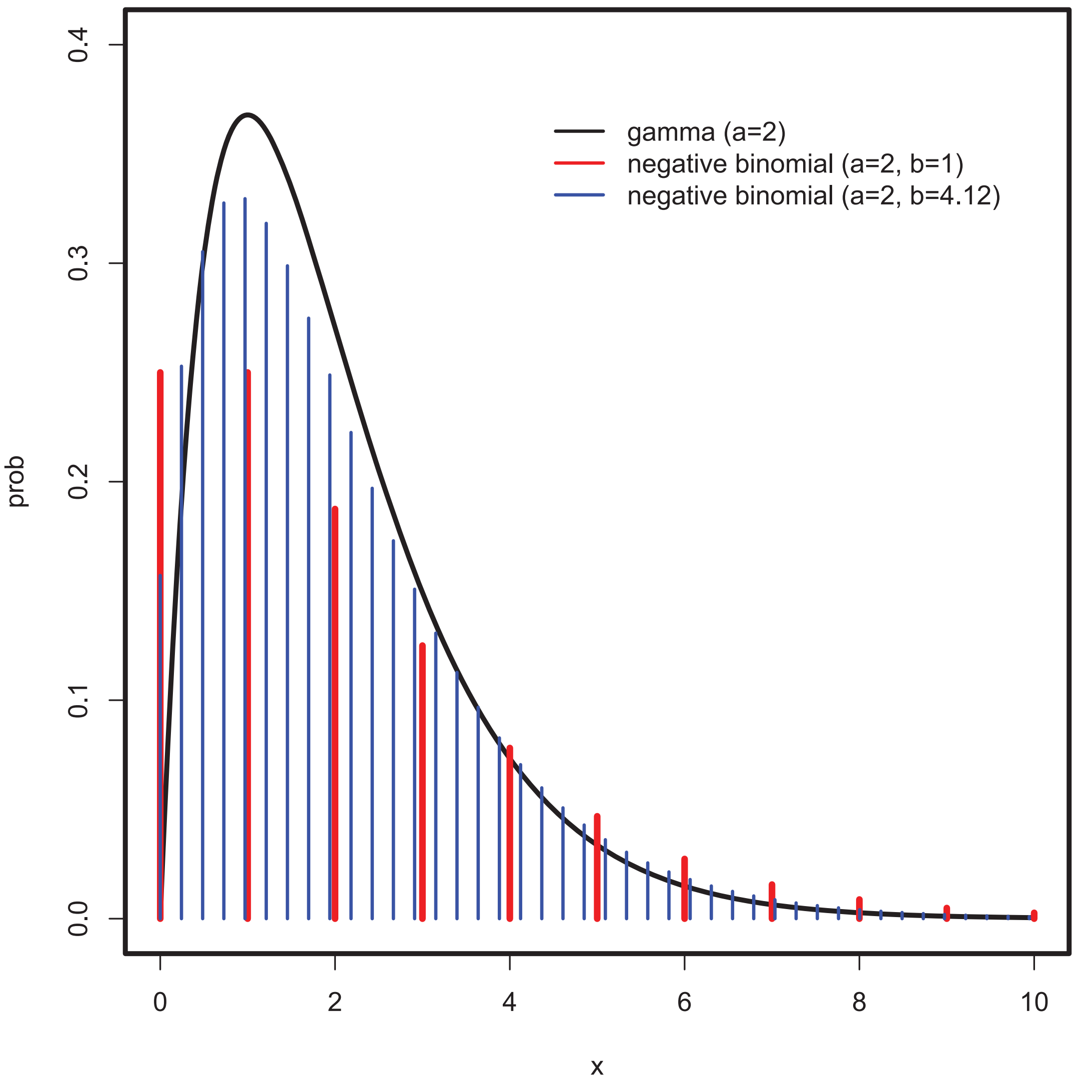
$$\phi(\theta) = \det(I - \Theta\Sigma)^{-\alpha} e^{\beta \mathrm{tr}(I-\Theta\Sigma)^{-1}\Theta\Delta}$$

  and

$$\psi_b(z) = \text{const}\,\det(I - ZQ)^{-\alpha} e^{\beta \mathrm{tr}(I-ZQ)^{-1}ZL}$$

  $(Q = I + (I+\Sigma)^{-1}$, $L = (I+\Sigma)^{-1}\Delta(I+\Sigma)^{-1})$ share the same form!

- By the moment formula of Wishart, we can confirm that $\psi_b(z)$ is the pgf of a probability measure under the assumption of Theorem



Convergence of $\psi_b(e^{\theta/b}) \longrightarrow \phi(\theta)$ as $b \to \infty$

— $b = 1$

— $b = 4.12$

— $b = \infty$

Summary

1. **Expressions for the moments of noncentral Wishart matrices**

$$\boldsymbol{E}[w_{i_1 i_2} \cdots w_{i_{2n-1} i_{2n}}]$$

are given in terms of graph terminology

2. **As an example, a condition for "infinite divisibility" of the noncentral multivariate gamma distribution is given**